

On reflected Stratonovich stochastic differential equations

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Abstract

We study the problem of existence, uniqueness and approximation of solutions of finite dimensional Stratonovich stochastic differential equations with reflecting boundary condition driven by semimartingales with jumps. As an application we generalize known results on the Wong-Zakai type approximations.

Key Words: reflected stochastic differential equation, Stratonovich stochastic differential equation, Wong-Zakai type approximation.

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1 Introduction

Let D be a connected domain in \mathbb{R}^d . We consider a d -dimensional stochastic differential equations (SDEs) on a domain D with reflecting boundary condition of the form

$$X_t = X_0 + \int_0^t f(X_s) \circ dZ_s + K_t, \quad t \in \mathbb{R}^+. \quad (1.1)$$

Here the notation ‘ \circ ’ indicates that we deal with stochastic integral of Stratonovich type. In (1.1) $X_0 \in \bar{D} = D \cup \partial D$, X is a reflecting process on \bar{D} , K is a bounded variation process with variation $|K|$ increasing only when $X_t \in \partial D$ and Z is a d -dimensional semimartingale with jumps.

In the nonreflected case, that is, when $D = \mathbb{R}^d$ and $K = 0$ the above type of SDEs has been investigated in many papers (see, e.g., [12, 13, 15, 16, 17, 26, 29, 32, 33]). In particular, Wong and Zakai [32, 33] proved that for Z being a Brownian motion the solution of (1.1) is a limit of the following simple approximation scheme: $\hat{X}_0^n = X_0$,

$$\hat{X}_t^n = \hat{X}_{\frac{k}{n}}^n + n \int_{\frac{k}{n}}^t f(\hat{X}_s^n) ds (Z_{\frac{k+1}{n}} - Z_{\frac{k}{n}}), \quad t \in (\frac{k}{n}, \frac{k+1}{n}], \quad k \in \mathbb{N} \cup \{0\}.$$

Since that time approximations of the above type are called Wong-Zakai approximations. The case of nonreflected Stratonovich SDEs driven by semimartingales with jumps has been

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studied by Mackevicius [15], Marcus [16, 17] and in the general case by Kurtz, Pardoux and Protter [13]. Stratonovich SDEs considered in the last paper has the following form:

$$X_t = X_0 + \int_0^t f(X_s) \circ dZ_s = X_0 + \int_0^t f(X_{s-}) dZ_s + \frac{1}{2} \int_0^t f' f(X_{s-}) d[Z]_s^c + \sum_{s \leq t} \{ \varphi(f \Delta Z_s, X_{s-}) - X_{s-} - f(X_{s-}) \Delta Z_s \},$$

where for given $g \in \mathcal{C}^1(\mathbb{R}^d, \mathbb{R}^d)$, $\varphi(g, x)$ denotes the value at time $u = 1$ of the solution of the ordinary differential equation $\frac{dy}{du}(u) = g(y(u))$, $y(0) = x \in \mathbb{R}^d$. In particular, Kurtz, Pardoux and Protter were able to prove that $\hat{X}_t^n \rightarrow_{\mathcal{P}} X_t$ except possibly for a countable set of t 's. This convergence cannot be in general strengthened to the convergence in the Skorokhod topology J_1 (in contrast to the approximating sequence $\{\hat{X}^n\}$, the solution X need not have continuous trajectories).

Reflected Stratonovich SDEs has been studied for the first time in Doss and Priouret [2]. In [2] the convergence of Wong–Zakai type approximations has been proved in the case of diffusion processes and sufficiently smooth ∂D . Next, the almost sure convergence of Wong–Zakai approximation has been observed by Pettersson [21] in the case of convex domain and constant diffusion coefficient. Pettersson's results have been refined by Ren and Xu [23, 24] in papers devoted to multivalued SDEs, which correspond to SDEs with reflecting boundary conditions on convex domains. Quite recently, Evans and Stroock [4] have proved weak convergence of Wong–Zakai approximations in the case of not necessary smooth domains satisfying conditions (A), (B) and (C) from the paper by Lions and Sznitman [14]. Their result has been strengthened to L^p convergence by Aida and Sasaki [1]. The Wong–Zakai approximation of reflected diffusion processes has also been studied by Zhang [34, 35]. We emphasize that in all the above mentioned papers the limit of Wong–Zakai approximations is a continuous solution to some reflected Stratonovich SDE.

In the present paper we remove condition (C) from [14] and we only assume that D satisfies conditions (A), (B). We study the general reflected Stratonovich SDEs driven by semimartingales with jumps (1.1) and various methods of its approximations including Wong–Zakai type approximations. Since the Stratonovich stochastic integral considered in this paper coincides with the general Stratonovich stochastic integral studied in the paper by Kurtz, Pardoux and Protter, our results generalize appropriate results from the above mentioned papers. The problem of existence and uniqueness of solutions to reflected Stratonovich SDEs with jumps in domains satisfying conditions (A), (B) has been also considered by Kohatsu-Higa [9] but the Stratonovich integral in [9] is different from our integral. In [9] the integral has the property that its jumps place the solution inside the domain. This implies that the compensating reflection process K has continuous trajectories and its variation is increasing only when the solution X is living on the boundary in a continuous manner. Using very subtle and difficult methods based on the multidimensional change of time (see, e.g., [10]) Kohatsu-Higa was able to prove the existence and uniqueness results for such reflected SDEs, provided that the driving semimartingale Z has summable jumps, i.e. $\sum_{s \leq t} |\Delta Z_s| < \infty$, P -a.s., $t \in \mathbb{R}^+$.

The paper is organized as follows.

Section 2 presents some preliminaries concerning solutions of the Skorokhod problem and solutions of reflected SDEs.

In Section 3 we study stability of reflected SDEs of the form (1.1). More precisely, we consider a sequence of semimartingales $\{Z^n\}$ satisfying the condition (UT) and a sequence

$\{(X^n, K^n)\}$ of solutions of SDEs of the form (1.1), i.e.

$$X_t^n = X_0^n + \int_0^t f(X_s^n) \circ dZ_s^n + K_t^n, \quad t \in \mathbb{R}^+.$$

We give conditions ensuring weak and strong convergence of $\{(X^n, K^n)\}$ to the solution (X, K) of (1.1). Consequently, we get the existence of weak solution of (1.1) provided that $f, f'f$ are continuous and bounded and $|\Delta Z| < r_0 / \sup_{x \in \bar{D}} \|f(x)\|$, where r_0 is some constant depending on the domain D . If additionally $f, f'f$ are locally Lipschitz continuous, we prove the existence and uniqueness of strong solutions to (1.1).

Section 4 is devoted to the study of Wong-Zakai type approximations of the unique strong solution (X, K) of (1.1). We consider two approximation schemes. The first one is defined by the recurrent formula: $\hat{X}_0^n = X_0$ and

$$\hat{X}_t^n = \Pi_{\bar{D}}(\hat{X}_{\frac{k}{n}}^n) + n \int_{\frac{k}{n}}^t f(\hat{X}_s^n) ds (Z_{\frac{k+1}{n}} - Z_{\frac{k}{n}}), \quad t \in [\frac{k}{n}, \frac{k+1}{n}), \quad k \in \mathbb{N} \cup \{0\},$$

where $\Pi_{\bar{D}}(x)$ is the projection of x on \bar{D} . Applying the approximation results from Section 3 we show that $\hat{X}^n \rightarrow_{\mathcal{P}} X$ in the S topology introduced by Jakubowski [7], and that $\hat{X}_t^n \rightarrow_{\mathcal{P}} X_t$ provided that $\Delta Z_t = 0$, $t \in \mathbb{R}^+$. The S topology is weaker than the Skorokhod topology J_1 but stronger than the Meyer-Zheng topology (see, e.g., [10, 19]). In the general case our convergence results cannot be strengthened to the convergence in the Skorokhod topology J_1 . However, in case Z has continuous trajectories we prove that $\sup_{t \leq q} |\hat{X}_t^n - X_t| \rightarrow_{\mathcal{P}} 0$, $q \in \mathbb{R}^+$. The second scheme has the following form: $\bar{X}_0^n = X_0$ and

$$\bar{X}_t^n = \bar{X}_{\frac{k}{n}}^n + n \int_{\frac{k}{n}}^t f(\bar{X}_s^n) ds (Z_{\frac{k+1}{n}} - Z_{\frac{k}{n}}) + \bar{K}_t^n - \bar{K}_{\frac{k}{n}}^n, \quad t \in [\frac{k}{n}, \frac{k+1}{n}), \quad k \in \mathbb{N} \cup \{0\},$$

where in each step some appropriate deterministic reflected differential equation is solved. This is well known method of approximation of reflected diffusions (see, e.g., [1, 2, 4, 21, 23, 24]). We show that for any continuous semimartingale Z , $\sup_{t \leq q} |\bar{X}_t^n - X_t| \rightarrow_{\mathcal{P}} 0$, $q \in \mathbb{R}^+$. Unfortunately, this method is not applicable in case of equations of the form (1.1) driven by semimartingales with jumps (see Remark 4.3).

We will use the following notation. $\mathbb{R}^+ = [0, \infty)$. $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ is the space of all càdlàg mappings $x : \mathbb{R}^+ \rightarrow \mathbb{R}^d$, i.e. mappings which are right continuous and admit left-hands limits equipped with the Skorokhod J_1 topology. For $x \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$, $t > 0$, we write $x_{t-} = \lim_{s \uparrow t} x_s$, $\Delta x_t = x_t - x_{t-}$. For x with locally bounded variation we denote by $|x|_t$ its total variation on the interval $[0, t]$, i.e., $|x|_t = \sup_{\pi} \sum_{j=1}^n |x_{t_j} - x_{t_{j-1}}| < +\infty$, where the supremum is taken over all subdivisions $\pi = \{0 = t_0 < \dots < t_n = t\}$ of $[0, t]$ and $|\cdot|$ denotes the usual Euclidean norm in \mathbb{R}^d . Every process Z appearing in the sequel is assumed to have càdlàg trajectories. If $Z = (Z^1, \dots, Z^d)$ is a semimartingale then $[Z]_t$ stands for $\sum_{i=1}^d [Z^i]_t$ and $[Z^i]$ stands for the quadratic variation process of Z^i , $i = 1, \dots, d$. Similarly, $\langle Z \rangle_t = \sum_{i=1}^d \langle Z^i \rangle_t$ and $\langle Z^i \rangle$ stands for the predictable compensator of $[Z^i]$, $i = 1, \dots, d$. " $\rightarrow_{\mathcal{D}}$ ", " $\rightarrow_{\mathcal{P}}$ " denote convergence in law and in probability, respectively.

2 Preliminaries

Let D be a nonempty connected domain in \mathbb{R}^d . We define the set \mathcal{N}_x of inward normal unit vectors at $x \in \partial D$ by

$$\mathcal{N}_x = \bigcup_{r>0} \mathcal{N}_{x,r}, \quad \mathcal{N}_{x,r} = \{\mathbf{n} \in \mathbb{R}^d; |\mathbf{n}| = 1, B(x - r\mathbf{n}, r) \cap D = \emptyset\},$$

where $B(z, r) = \{y \in \mathbb{R}^d; |y - z| < r\}$, $z \in \mathbb{R}^d$, $r > 0$. Following Lions and Sznitman [14] and Saisho [25] we consider two assumptions.

(A) There exists a constant $r_0 > 0$ such that

$$\mathcal{N}_x = \mathcal{N}_{x,r_0} \neq \emptyset \quad \text{for every } x \in \partial D.$$

(B) There exist constants $\delta > 0$, $\beta \geq 1$ such that for every $x \in \partial D$ there exists a unit vector \mathbf{l}_x with the following property

$$\langle \mathbf{l}_x, \mathbf{n} \rangle \geq \frac{1}{\beta} \quad \text{for every } \mathbf{n} \in \bigcup_{y \in B(x, \delta) \cap \partial D} \mathcal{N}_y$$

where $\langle \cdot, \cdot \rangle$ denotes the usual inner product in \mathbb{R}^d .

Remark 2.1 ([14, 25, 31])

- (i) $\mathbf{n} \in \mathcal{N}_{x,r}$ if and only if $\langle y - x, \mathbf{n} \rangle + \frac{1}{2r}|y - x|^2 \geq 0$ for every $y \in \bar{D}$.
- (ii) If $\text{dist}(x, \bar{D}) < r_0$, $x \notin \bar{D}$ then there exists a unique $\Pi_{\bar{D}}(x) \in \bar{D}$ such that $|x - \Pi_{\bar{D}}(x)| = \text{dist}(x, \bar{D})$. Moreover, $(\Pi_{\bar{D}}(x) - x)/|\Pi_{\bar{D}}(x) - x| \in \mathcal{N}_{\Pi_{\bar{D}}(x)}$.
- (iii) If D is a convex domain in \mathbb{R}^d then $r_0 = +\infty$.

Let $y \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ and $y_0 \in \bar{D}$. We recall that a pair $(x, k) \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d})$ is a solution of the Skorokhod problem associated with y if

- $x_t = y_t + k_t$, $t \in \mathbb{R}^+$,
- $x_t \in \bar{D}$, $t \in \mathbb{R}^+$,
- k is a function with bounded variation on each finite interval such that $k_0 = 0$ and

$$k_t = \int_0^t \mathbf{n}_s d|k|_s, \quad |k|_t = \int_0^t \mathbf{1}_{\{x_s \in \partial D\}} d|k|_s,$$

where $\mathbf{n}_s \in \mathcal{N}_{x_s}$ if $x_s \in \partial D$.

The problem of existence and approximation of solutions of the Skorokhod problem in domains satisfying conditions (A) and (B) has been studied by Saisho [25] (the case of continuous functions) and Słomiński [27] (the case of càdlàg functions). We now recall the general approximation method considered in these papers.

Let $\{\{t_k^n\}\}$ be an array of nonnegative numbers such that in the n^{th} row the sequence $T_n = \{t_k^n\}$ forms a partition of \mathbb{R}^+ such that $0 = t_0^n < t_1^n < \dots$, $\lim_{k \rightarrow \infty} t_k^n = +\infty$ and

$$\max_k (t_k^n - t_{k-1}^n) \longrightarrow 0 \quad \text{as } n \rightarrow +\infty. \quad (2.1)$$

Given $\{t_k^n\}$ we define a sequence of summation rules $\{\rho^n\}$, $\rho^n : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ by $\rho_t^n = \max\{t_k^n; t_k^n \leq t\}$. For every $y \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ the sequence $\{y^{\rho^n}\}$ denotes the following discretizations of y :

$$y_t^{\rho^n} = y_{t_k^n} = y_{t_k^n} \quad \text{for } t \in [t_k^n, t_{k+1}^n), k \in \mathbb{N} \cup \{0\}, n \in \mathbb{N}.$$

Using for instance [3, Proposition 3.6.5] one can check that $y^{\rho^n} \rightarrow y$ in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$. From [27, Corollary 3] it follows that if (A) and (B) are satisfied then for every $y \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ such that $y_0 \in \bar{D}$ and $|\Delta y| < r_0$ there exists a unique solution (x, k) of the Skorokhod problem associated with y . Moreover, if $\{(x^n, k^n)\}$ is the sequence of solutions of the Skorokhod problem associated with the sequence of discretizations $\{y^{\rho^n}\}$ then

$$(x^n, k^n, y^{\rho^n}) \rightarrow (x, k, y) \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{3d}). \quad (2.2)$$

It is worth noting that for all sufficiently large n , (x^n, k^n) are defined by the following recurrent formula: $x_0^n = y_0$, $k_0^n = 0$,

$$\begin{cases} x_{t_{k+1}^n}^n &= \Pi_{\bar{D}}(x_{t_k^n}^n + (y_{t_{k+1}^n}^n - y_{t_k^n}^n)), \\ k_{t_{k+1}^n}^n &= k_{t_k^n}^n + (x_{t_{k+1}^n}^n - x_{t_k^n}^n) - (y_{t_{k+1}^n}^n - y_{t_k^n}^n) \end{cases}$$

and $x_t^n = x_{t_k^n}^n$, $k_t^n = k_{t_k^n}^n$, $t \in [t_k^n, t_{k+1}^n)$, $k \in \mathbb{N} \cup \{0\}$.

Lemma 2.2 *Assume (A) and (B). Let $y \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ be such that $y_0 \in \bar{D}$, $|\Delta y| < r_0$, and let (x, k) denote the solution of the Skorokhod problem associated with y . Then for every $0 \leq t < q$*

$$|k|_{[t,q]} \leq |y|_{[t,q]} \quad \text{and} \quad |x|_{[t,q]} \leq 2|y|_{[t,q]}.$$

PROOF. Without loss of generality we may assume that $t = 0$. Let $\{(x^n, k^n)\}$ be a sequence of solutions of the Skorokhod problem associated with the sequence of discretizations $\{y^{\rho^n}\}$. Clearly,

$$|\Delta k_{t_{k+1}^n}^n| = |k_{t_{k+1}^n}^n - k_{t_k^n}^n| \leq |y_{t_{k+1}^n}^n - y_{t_k^n}^n| = |\Delta y_{t_{k+1}^n}^{\rho^n}|, \quad k \in \mathbb{N} \cup \{0\},$$

which implies that for every $q \in \mathbb{R}^+$,

$$|k^n|_q \leq \sum_{k; t_{k+1}^n \leq q} |k_{t_{k+1}^n}^n - k_{t_k^n}^n| \leq \sum_{k; t_{k+1}^n \leq q} |y_{t_{k+1}^n}^n - y_{t_k^n}^n| = |y^{\rho^n}|_q.$$

Since $k^n \rightarrow k$ in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$,

$$|k|_q \leq \liminf_{n \rightarrow \infty} |k^n|_q \leq \sup_n |y^{\rho^n}|_q \leq |y|_q, \quad \text{provided that } \Delta y_q = 0.$$

If $\Delta y_q \neq 0$ then there exists a sequence $\{q_i\}$ such that $q_i \downarrow q$ and $\Delta y_{q_i} = 0$, $i \in \mathbb{N}$. Then $|k|_{q_i} \leq |y|_{q_i}$, $i \in \mathbb{N}$, so letting $i \rightarrow \infty$ we obtain the desired result. \square

Lemma 2.2 is a generalization of [14, Theorem 2.1], where the case of continuous y and smooth domains is considered. In [1] estimates of Lemma 2.2 but with greater constants were proved for continuous y and domains satisfying condition (A) only.

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space and let (\mathcal{F}_t) be a filtration on $(\Omega, \mathcal{F}, \mathcal{P})$ satisfying the usual conditions. Let Y be an (\mathcal{F}_t) adapted process and $Y_0 \in \bar{D}$. We say that a pair (X, K) of (\mathcal{F}_t) adapted processes solves the Skorokhod problem associated with Y if and only

if for every $\omega \in \Omega$, $(X(\omega), K(\omega))$ is a solution of the Skorokhod problem corresponding to $Y(\omega)$. Let us note that by [27, Corollary 6.10] for every process Y such that $Y_0 \in \bar{D}$ and $|\Delta Y| < r_0$ there exists a unique solution of the Skorokhod problem associated with Y .

We will consider processes Y, Y' admitting the decompositions

$$Y_t = H_t + M_t + V_t, \quad Y'_t = H_t + M'_t + V'_t, \quad t \in \mathbb{R}^+, \quad (2.3)$$

where H is an (\mathcal{F}_t) adapted process, M, M' are (\mathcal{F}_t) adapted local martingale with $M_0 = M'_0 = 0$ and V, V' are (\mathcal{F}_t) adapted processes of bounded variation with $V_0 = V'_0 = 0$.

Remark 2.3 ([27, 28]) Assume that Y, Y' admit decompositions (2.3) and $Y_0, Y'_0 \in \bar{D}$. Let $(X, K), (X', K')$ denote solutions of the Skorokhod problems associated with processes Y and Y' , respectively. If $|\Delta Y|, |\Delta Y'| \leq \frac{r_0}{4}$ and there exists a constant a such that $|K|_\infty, |K'|_\infty \leq a$ (in the case $r_0 < \infty$) then for every $p \in \mathbb{N}$ there exists a constant $C(p)$ depending on p (and also on a, r_0, β, δ) such that

$$E \sup_{t \leq \tau} |X_t - X'_t|^{2p} \leq C(p) E([M - M']_\tau^p + |V - V'|_\tau^{2p}) \quad (2.4)$$

and

$$E \sup_{t < \tau} |X_t - X'_t|^{2p} \leq C(p) E([M - M']_{\tau-}^p + |V - V'|_{\tau-}^{2p} + \langle M - M' \rangle_{\tau-}^p) \quad (2.5)$$

for every stopping time τ .

Let $X_0 \in \bar{D}$ and let $Z_t = (Z_t^1, \dots, Z_t^d)$ be an (\mathcal{F}_t) adapted semimartingale such that $Z_0 = 0$. Given $f : \bar{D} \rightarrow \mathbb{R}^d \otimes \mathbb{R}^d$, $f(x) = \{f_{ik}\}_{i,k=1,\dots,d}$ such that f belongs to $\mathcal{C}^1(\mathbb{R}^d, \mathbb{R}^d \otimes \mathbb{R}^d)$ we consider reflected Stratonovich SDE of the form

$$\begin{aligned} X_t &= X_0 + \int_0^t f(X_s) \circ dZ_s + K_t \\ &= X_0 + \int_0^t f(X_{s-}) dZ_s + \frac{1}{2} \int_0^t f' f(X_{s-}) d[Z]_s^c \\ &\quad + \sum_{s \leq t} \{\varphi(f \Delta Z_s, X_{s-}) - X_{s-} - f(X_{s-}) \Delta Z_s\} + K_t, \end{aligned} \quad (2.6)$$

where for given $g \in \mathcal{C}^1(\mathbb{R}^d, \mathbb{R}^d)$, $\varphi(g, x)$ denotes the value at time $u = 1$ of the solution of the following ordinary differential equation:

$$\frac{dy}{du}(u) = g(y(u)); \quad y(0) = x \in \mathbb{R}^d.$$

Equivalently, for $i = 1, \dots, d$,

$$\begin{aligned} X_t^i &= X_0^i + \sum_{j=1}^d \int_0^t f_{ij}(X_{s-}) dZ_s^j + \frac{1}{2} \sum_{j=1}^d \sum_{l=1}^d \sum_{m=1}^d \int_0^t \frac{\partial f_{ij}}{\partial x_l} f_{lm}(X_{s-}) d[Z^j, Z^m]_s^c \\ &\quad + \sum_{s \leq t} \{\varphi^i(f \Delta Z_s, X_{s-}) - X_{s-}^i - \sum_{j=1}^d f_{ij}(X_{s-}) \Delta Z_s^j\} + K_t^i, \quad t \in \mathbb{R}^+. \end{aligned}$$

Note that in case Z has continuous trajectories the definition of Stratonovich stochastic integral given above coincides with the well known definition given by Meyer [20].

We say that the SDE (2.5) has a strong solution if there exists a pair (X, K) of (\mathcal{F}_t) adapted processes which solves the Skorokhod problem associated with

$$Y_t = X_0 + \int_0^t f(X_s) \circ dZ_s, \quad t \in \mathbb{R}^+.$$

In the sequel we say that $f, f'f$ have some property if the coefficients $f_{ij}, \frac{\partial f_{ij}}{\partial x_l} f_{lm}$ have this property for $i, j, l, m = 1, \dots, d$.

Remark 2.4 (a) The existence and uniqueness of strong solutions of usual SDEs driven by semimartingales with reflecting boundary on domains satisfying conditions (A) and (B) has been studied in detail in [27]. In particular, in [27, Theorem 5] it is proved that if f is bounded and Lipschitz continuous, $X_0 \in \bar{D}$ and $|\Delta Z| < r_0/L$, where $L = \sup_{x \in \bar{D}} \|f(x)\|$, then there exists a unique strong solution of the reflecting SDE of the form

$$X_t = X_0 + \int_0^t f(X_{s-}) dZ_s + K_t, \quad t \in \mathbb{R}^+.$$

Since (2.6) has the additional term $\sum_{s \leq t} \{\varphi(f\Delta Z_s, X_{s-}) - X_{s-} - f(X_{s-})\Delta Z_s\}$, $t \in \mathbb{R}^+$, [27, Theorem 5] does not apply to Stratonovich SDE with reflection.

(b) If $f'f$ is bounded then the results proved in [13, page 356] there is $C > 0$ such that for every $s \in \mathbb{R}^+$ we have $|\varphi(f\Delta Z_s, X_{s-}) - X_{s-} - f(X_{s-})\Delta Z_s| \leq C|\Delta Z_s|^2$. Hence

$$\sum_{s \leq t} |\varphi(f\Delta Z_s, X_{s-}) - X_{s-} - f(X_{s-})\Delta Z_s| \leq C \sum_{s \leq t} |\Delta Z_s|^2, \quad t \in \mathbb{R}^+,$$

which means that the additional term is finite a.s. One can observe that as in the nonreflected case considered in [13], (X, K) is a strong solution of (2.6) iff it satisfies

$$\begin{aligned} X_t &= X_0 + \int_0^t f(X_{s-}) dZ_s + \frac{1}{2} \int_0^t f'f(X_{s-}) d[Z]_s^c \\ &\quad + \int_0^t h(s, \cdot, X_{s-}) d[Z]_s^d + K_t, \quad t \in \mathbb{R}^+, \end{aligned} \tag{2.7}$$

where $h(s, \omega, x) = (\varphi(f\Delta Z_s, x) - x - f(x)\Delta Z_s)/|\Delta Z_s|^2$, $s \in \mathbb{R}^+$, $x \in \bar{D}$ is bounded by the constant C . Moreover, if $f, f'f$ are Lipschitz continuous and the jumps of Z are bounded, then there is another constant C' such that

$$|h(s, \omega, x) - h(s, \omega, y)| \leq C'|x - y|, \quad x, y \in \bar{D}, \omega \in \Omega. \tag{2.8}$$

(see [13, Lemma 3.1]).

We say that (2.6) has the pathwise uniqueness property if for any probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})$ with filtration $(\bar{\mathcal{F}}_t)$ and any $\bar{X}_0 \in \bar{D}$ and $(\bar{\mathcal{F}}_t)$ -adapted semimartingale \bar{Z} such that $\mathcal{L}(\bar{X}_0, \bar{Z}) = \mathcal{L}(X_0, Z)$, $P[(\bar{X}_t, \bar{K}_t) = (\bar{X}'_t, \bar{K}'_t), t \in \mathbb{R}^+] = 1$ for any two $(\bar{\mathcal{F}}_t)$ adapted strong solutions $(\bar{X}, \bar{K}), (\bar{X}', \bar{K}')$ of (2.6) on $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})$.

We will consider the following assumptions.

(F) $f \in \mathcal{C}^1(\mathbb{R}^d, \mathbb{R}^d \otimes \mathbb{R}^d)$, $f, f'f$ are bounded.

(Δ) $|\Delta Z| < r_0 / \sup_{x \in \bar{D}} \|f(x)\|$.

Lemma 2.5 *Assume (A),(B) and (F),(Δ). If $f, f'f$ are locally Lipschitz continuous, then the SDE (2.6) has the pathwise uniqueness property.*

PROOF. Without loss of generality we may consider solutions of (2.6) on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ with a filtration (\mathcal{F}_t) . Since Z has bounded jumps, it is a special semimartingale. Therefore Z admits decomposition of the form $Z_t = M_t + V_t$, $t \in \mathbb{R}^+$, where M is an (\mathcal{F}_t) adapted local martingale and V is an (\mathcal{F}_t) predictable processes of bounded variation such that $M_0 = V_0 = 0$ and $|\Delta V_t| \leq r_0/L$, $|\Delta M_t| \leq (2r_0)/L$, where $L = \sup_{x \in \bar{D}} \|f(x)\|$. Let (X, K) , (X', K') be two solutions of (2.6). Set

$$\tau_a = \inf\{t; \max(|X_t|, |X'_t|, |K_t|, |K'_t|, [M]_t, \langle M \rangle_t, |V|_t) > a\}, \quad a \in \mathbb{N}.$$

We will show that $(X, K) = (X', K')$ on any interval $[0, \tau_a]$, $a \in \mathbb{N}$. Using the fact that $|\Delta \int_0^t f(X_s) \circ dZ_s| < r_0$ there is a constant $C_a > 0$ such that $\sup_{t \leq \tau_a} |X_t|, \sup_{t \leq \tau_a} |X'_t|, |K|_{\tau_a}, |K'|_{\tau_a}, [M]_{\tau_a}, \langle M \rangle_{\tau_a}, |V|_{\tau_a} \leq C_a$. Moreover, $[Z]_{\tau_a} \leq 2[M]_{\tau_a} + 2|V|_{\tau_a}^2 \leq 2C_a + 2C_a^2$. By (2.5) and Lipschitz continuity of coefficients on $B(0, a)$, for any stopping time σ we have

$$\begin{aligned} E \sup_{t < \sigma} |X_t^{\tau_a} - X_t'^{\tau_a}|^2 &\leq C(2, a) E \left(\left| \int_0^\cdot (f(X_{s-}) - f(X'_{s-})) dM_s^{\tau_a} \right|_{\sigma-} \right. \\ &\quad + \left| \int_0^\cdot (f(X_{s-}) - f(X'_{s-})) dM_s^{\tau_a} \right|_{\sigma-} + \left| \int_0^\cdot (f(X_{s-}) - f(X'_{s-})) dV_s^{\tau_a} \right|_{\sigma-} \\ &\quad + \left| \frac{1}{2} \int_0^\cdot (f'(f(X_{s-}) - f(X'_{s-}))) d[Z^{\tau_a}]_s^c \right|_{\sigma-} \\ &\quad + \left| \int_0^\cdot (h(s, \cdot, X_{s-}) - h(s, \cdot, X'_{s-})) d[Z^{\tau_a}]_s^d \right|_{\sigma-} \Big) \\ &\leq C(2, a) E \int_0^{\sigma-} \sup_{u < s} |X_u - X'_u| d([M^{\tau_a}] + \langle M^{\tau_a} \rangle + |V^{\tau_a}|^2)_s. \end{aligned}$$

Using this and the stochastic version of Gronwall's lemma (see, e.g. [15, Lemma 2] or [29, Lemma C1]) we conclude that $E \sup_t |X_t^{\tau_a} - X_t'^{\tau_a}|^2 = 0$, which implies that $(X, K) = (X', K')$ on $[0, \tau_a]$. Since $\tau_a \nearrow \infty$, P -a.s., the lemma follows. \square

3 Stability of reflected Stratonovich SDEs

Let $\{Z^n\}$ be a sequence of semimartingales defined possibly on different probability spaces $(\Omega^n, \mathcal{F}^n, \mathcal{P}^n)$ and adapted to different filtrations (\mathcal{F}_t^n) . We will assume that $\{Z^n\}$ satisfies condition (UT) introduced in Stricker [30], which have appeared to be useful in the theory of limit theorems for stochastic integrals and for solutions of SDEs (see, e.g., [8, 11, 18, 26, 27]). We recall that $\{Z^n\}$ satisfies (UT) if

(UT) for every $q \in \mathbb{R}^+$ the family of random variables

$$\left\{ \int_{[0, q]} V_s^n dZ_s^n; n \in \mathbb{N}, V^n \in \mathbf{V}_q^n \right\} \quad \text{is tight in } \mathbb{R},$$

where \mathbf{V}_q^n is the class of all discrete predictable processes of the form $V_s^n = V_0^n + \sum_{i=0}^k V_i^n \mathbf{1}_{\{t_i < s \leq t_{i+1}\}}$ such that $0 = t_0 < t_1 < \dots < t_k = q$ and every V_i^n is $\mathcal{F}_{t_i}^n$ measurable, $|V_i^n| \leq 1$ for every $i \in \mathbb{N} \cup \{0\}$, $n \in \mathbb{N}$, $k \in \mathbb{N}$.

Let $\{(X^n, K^n)\}$ be a sequence of strong solutions of (2.6) driven by $\{Z^n\}$, i.e.

$$\begin{aligned} X_t^n &= X_0^n + \int_0^t f(X_s^n) \circ dZ_s^n + K_t^n \\ &= X_0^n + \int_0^t f(X_{s-}^n) dZ_s^n + \frac{1}{2} \int_0^t f' f(X_{s-}^n) d[Z^n]_s^c \\ &\quad + \sum_{s \leq t} \{\varphi(f \Delta Z_s^n, X_{s-}^n) - X_{s-}^n - f(X_{s-}^n) \Delta Z_s^n\} + K_t^n, \quad t \in \mathbb{R}^+. \end{aligned} \quad (3.1)$$

We say that the SDE (2.6) has a weak solution if there exists a probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathcal{P}})$ with filtration $(\hat{\mathcal{F}}_t)$ satisfying the usual conditions and an $(\hat{\mathcal{F}}_t)$ adapted processes $\hat{X}, \hat{K}, \hat{Z}$ such that $\mathcal{L}(\hat{X}_0, \hat{Z}) = \mathcal{L}(X_0, Z)$ and (2.6) holds for processes $\hat{X}, \hat{K}, \hat{Z}$ instead of X, K, Z . If $\mathcal{L}((\hat{X}, \hat{K})) = \mathcal{L}((\hat{X}', \hat{K}'))$ for any two weak solutions $(\hat{X}, \hat{K}), (\hat{X}', \hat{K}')$ of (2.6), possibly defined on two different probability spaces, then we say that (2.6) has the weak uniqueness property.

Theorem 3.1 *Assume (A),(B) and (F), (Δ) . Let $\{Z^n\}$ be a sequence of \mathcal{F}^n adapted semimartingales satisfying (UT), $\{X_0^n\}$ be a sequence of \mathcal{F}_0^n measurable random variables such that $X_0^n \in \bar{D}$ for $n \in \mathbb{N}$ and let*

$$(X_0^n, Z^n) \xrightarrow{\mathcal{D}} (X_0, Z) \quad \text{in } \mathbb{R}^d \times \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d).$$

If $\{(X^n, K^n)\}$ is a sequence of strong solutions of the SDE (3.1), then

- (i) $\{X^n\}$ satisfies (UT) and $\{|K^n|_q\}$ is bounded in probability, $q \in \mathbb{R}^+$,
- (ii) $\{(X^n, K^n)\}$ is tight in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d})$ and its every limit point is a weak solution of (2.6),
- (iii) if moreover (2.6) has the weak uniqueness property and (X, K) is its weak solution then

$$(X^n, K^n) \xrightarrow{\mathcal{D}} (X, K) \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d}).$$

PROOF. We follow the proof of [27, Theorem 4] and [29, Theorem 5.2].

(i) Set

$$h^n(s, x) = \frac{\varphi(\Delta Z_s^n f, x) - x - f(x) \Delta Z_s^n}{|\Delta Z_s^n|^2}, \quad (s, x) \in \mathbb{R}^+ \times \bar{D}, \quad n \in \mathbb{N}$$

and observe that (3.1) can be rewritten into the form

$$\begin{aligned} X_t^n &= X_0^n + \int_0^t f(X_{s-}^n) dZ_s^n + \frac{1}{2} \int_0^t f' f(X_{s-}^n) d[Z^n]_s^c \\ &\quad + \int_0^t h^n(s, X_{s-}^n) d[Z^n]_s^d + K_t^n. \end{aligned}$$

By Remark 2.4(b), $|h^n| \leq C$, $n \in \mathbb{N}$. Since $\{Z^n\}$ satisfies (UT), the sequence

$$\{Y^n = X_0^n + \int_0^\cdot f(X_{s-}^n) dZ_s^n + \frac{1}{2} \int_0^\cdot f' f(X_{s-}^n) d[Z^n]_s^c + \int_0^\cdot h^n(s, X_{s-}^n) d[Z^n]_s^d\}$$

satisfies (UT) as well (see, e.g., [18, Lemma 1.6]). Assertion (i) now follows from [27, Proposition 3].

(ii) By [6], $(Z^n, [[Z^n]]) \rightarrow_{\mathcal{D}} (Z, [[Z]])$ in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^{d(d+1)})$, which implies that

$$\{(Z^n, [[Z^n]]^c, [Z^n]^d, [[Z^n]])\} \text{ is tight in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{d(1+2d)+1}). \quad (3.2)$$

Using (3.2), part (i) and standard tightness criterions for sequences satisfying (UT) (see, e.g., [18, Proposition 3.3] one can show that $\{(Y^n, Z^n, [[Z^n]])\}$ is tight in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^{d(2+d)})$. From this and [27, Proposition 4] we conclude that

$$\{(X^n, K^n, Z^n, [[Z^n]])\} \text{ is tight in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{d(3+d)}).$$

Without loss of generality we may assume that

$$(X^n, Z^n, K^n, [[Z^n]]) \xrightarrow{\mathcal{D}} (\hat{X}, \hat{K}, \hat{Z}, [[\hat{Z}]]) \text{ in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{d(3+d)}).$$

The proof of (ii) will be completed once we show that (\hat{X}, \hat{K}) is a solution of the reflecting SDE of the form

$$\begin{aligned} \hat{X}_t &= \hat{X}_0 + \int_0^t f(\hat{X}_{s-}) d\hat{Z}_s + \frac{1}{2} \int_0^t f'f(\hat{X}_{s-}) d[\hat{Z}]_s^c \\ &\quad + \int_0^t h(s, \hat{X}_{s-}) d[\hat{Z}]_s^d + \hat{K}_t, \quad t \in \mathbb{R}^+. \end{aligned}$$

while by [27, Proposition 4], to show the last statement it suffices to prove the convergence $(X^n, K^n, Y^n) \rightarrow_{\mathcal{D}} (\hat{X}, \hat{K}, \hat{Y})$ in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^{3d})$, where

$$\hat{Y}_t = \hat{X}_0 + \int_0^t f(\hat{X}_{s-}) d\hat{Z}_s + \frac{1}{2} \int_0^t f'f(\hat{X}_{s-}) d[\hat{Z}]_s^c + \int_0^t h(s, \hat{X}_{s-}) d[\hat{Z}]_s^d, \quad t \in \mathbb{R}^+. \quad (3.3)$$

Let $\{\epsilon_k\}$ be a sequence of constants such that $\epsilon_k \downarrow 0$ and $P[|\Delta Z_t| = \epsilon_k; t \in \mathbb{R}^+] = 0$, $k \in \mathbb{N}$. Set $J_t^{n, \epsilon_k} = \sum_{0 < s \leq t} \Delta Z_s^n \mathbf{1}_{\{|\Delta Z_s^n| > \epsilon_k\}}$, $\hat{J}_t^{\epsilon_k} = \sum_{0 < s \leq t} \Delta Z_s \mathbf{1}_{\{|\Delta \hat{Z}_s| > \epsilon_k\}}$, $Z^{n, \epsilon_k} = Z_t^n - J_t^{n, \epsilon_k}$ and $\hat{Z}^{\epsilon_k} = \hat{Z}_t - \hat{J}_t^{\epsilon_k}$, $k, n \in \mathbb{N}$. Since

$$(X^n, Z^n, [[Z^{n, \epsilon_k}]], [J^{n, \epsilon_k}]) \xrightarrow{\mathcal{D}} (\hat{X}, \hat{Z}, [[\hat{Z}^{\epsilon_k}]], [\hat{J}^{\epsilon_k}]) \text{ in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{d(2+d)+1}) \quad (3.4)$$

and $\sum_{s \leq t} |\Delta \hat{Z}_s|^2 \mathbf{1}_{\{|\Delta Z_s^n| > \epsilon_k\}} \nearrow [\hat{Z}]_t^d$, P -a.s. for $t \in \mathbb{R}^+$, we can find a sufficiently slowly increasing sequence $k_n \uparrow +\infty$ such that

$$(X^n, Z^n, [[Z^{n, \epsilon_{k_n}}]], [J^{n, \epsilon_{k_n}}]) \xrightarrow{\mathcal{D}} (\hat{X}, \hat{Z}, [[\hat{Z}]]^c, [\hat{Z}]^d) \text{ in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{d(2+d)+1}). \quad (3.5)$$

Set $\phi(y, x) = \varphi(fy, x) - x - f(x)y$, $x, y \in \bar{D}$ and observe that by (3.5),

$$\begin{aligned} &(X^n, K^n, Z^n, [[Z^{n, \epsilon_{k_n}}]], [J^{n, \epsilon_{k_n}}], \sum_{s \leq \cdot} \phi(\Delta Z_s^n, X_{s-}^n) \mathbf{1}_{\{|\Delta Z_s^n| > \epsilon_k\}}) \\ &\xrightarrow{\mathcal{D}} (\hat{X}, \hat{K}, \hat{Z}, [[\hat{Z}]]^c, [\hat{Z}]^d, \sum_{s \leq t} \phi(\Delta \hat{Z}_s, \hat{X}_{s-}) \mathbf{1}_{\{|\Delta \hat{Z}_s| > \epsilon_k\}}) \text{ in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{d(4+d)+1}). \end{aligned} \quad (3.6)$$

By Remark 2.4(b), $\sum_{s \leq t} |\phi(\Delta Z_s^n, X_{s-}^n)| \mathbf{1}_{\{\epsilon_k \geq |\Delta Z_s^n| > \epsilon_{k_n}\}} \leq C \sum_{s \leq t} |\Delta Z_s^n|^2 \mathbf{1}_{\{\epsilon_k \geq |\Delta Z_s^n|\}}$. Since

$$\sum_{s \leq \cdot} |\Delta Z_s^n|^2 \mathbf{1}_{\{\epsilon_k \geq |\Delta Z_s^n|\}} \xrightarrow{\mathcal{D}} \sum_{s \leq \cdot} |\Delta \hat{Z}_s|^2 \mathbf{1}_{\{\epsilon_k \geq |\Delta \hat{Z}_s|\}} \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R})$$

and $\sum_{s \leq t} |\Delta \hat{Z}_s|^2 \mathbf{1}_{\{\epsilon_k \geq |\Delta \hat{Z}_s|\}} \searrow 0$, P -a.s. for $t \in \mathbb{R}^+$, it follows from (3.6) that

$$\begin{aligned} & (X^n, K^n, Z^n, [[Z^{n, \epsilon_{k_n}}]], \sum_{s \leq \cdot} \phi(\Delta Z_s^n, X_{s-}^n) \mathbf{1}_{\{|\Delta Z_s^n| > \epsilon_{k_n}\}}) \\ & \xrightarrow{\mathcal{D}} (\hat{X}, \hat{K}, \hat{Z}, [[\hat{Z}]]^c, \sum_{s \leq \cdot} \phi(\Delta \hat{Z}_s, \hat{X}_{s-})) \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{d(4+d)}). \end{aligned} \quad (3.7)$$

By the above and [8, Theorem 2.6],

$$\begin{aligned} & (X^n, K^n, \int_0^\cdot f(X_{s-}^n) dZ_s^n, \int_0^\cdot f' f(X_{s-}^n) d[Z^{n, \epsilon_{k_n}}]_s, \sum_{s \leq \cdot} \phi(\Delta Z_s^n, X_{s-}^n) \mathbf{1}_{\{|\Delta Z_s^n| > \epsilon_{k_n}\}}) \\ & \xrightarrow{\mathcal{D}} (\hat{X}, \hat{K}, \int_0^\cdot f(\hat{X}_{s-}) d\hat{Z}_s, \int_0^\cdot f' f(\hat{X}_{s-}) d[\hat{Z}]^c, \sum_{s \leq \cdot} \phi(\Delta \hat{Z}_s, \hat{X}_{s-})) \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{4d}). \end{aligned}$$

Finally, arguing as in the proof of [29, Theorem 5.2] we show that

$$\frac{1}{2} \int_0^\cdot f' f(X_{s-}^n) d[Z^n]_s^c + \sum_{s \leq \cdot} \phi(\Delta Z_s^n, X_{s-}^n) \mathbf{1}_{\{|\Delta Z_s^n| \leq \epsilon_{k_n}\}} - \frac{1}{2} \int_0^\cdot f' f(X_{s-}^n) d[Z^{n, \epsilon_n}]_s \xrightarrow{\mathcal{P}} 0.$$

Consequently, $(X^n, Y^n, K^n) \xrightarrow{\mathcal{D}} (\hat{X}, \hat{Y}, \hat{K})$ in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^{3d})$, where \hat{Y} satisfies (3.3).

(iii) Follows immediately from (ii). \square

Corollary 3.2 *Assume (A), (B) and (F), (Δ) . Let Z be an (\mathcal{F}_t) adapted semimartingale and X_0 be \mathcal{F}_0 measurable random variable such that $X_0 \in \bar{D}$. Then there exists a weak solution of (2.6).*

PROOF. Let $\{T_n\}$ be a sequence of partitions of \mathbb{R}^+ satisfying (2.1). Set $\mathcal{F}_t^{\rho^n} = \mathcal{F}_{t_n}^{\rho^n}$. Then Z^{ρ^n} is an \mathcal{F}^{ρ^n} adapted semimartingale. By the Bichteler-Mokobodski theorem the sequence $\{Z^{\rho^n}\}$ satisfies (UT). Assume that $\{(X^n, K^n)\}$ is a family of strong solutions of (3.1) driven by $\{Z^{\rho^n}\}$, that is

$$\begin{aligned} X_t^n &= X_0 + \int_0^t f(X_{s-}^n) dZ_s^{\rho^n} + \frac{1}{2} \int_0^t f' f(X_{s-}^n) d[Z^{\rho^n}]_s^c \\ &\quad + \sum_{s \leq t} \{\varphi(f \Delta Z_s^{\rho^n}, X_{s-}^n) - X_{s-}^n - f(X_{s-}^n) \Delta Z_s^{\rho^n}\} + K_t^n \\ &= X_0 + \sum_{s \leq t} \{\varphi(f \Delta Z_s^{\rho^n}, X_{s-}^n) - X_{s-}^n\} + K_t^n, \quad t \in \mathbb{R}^+, n \in \mathbb{N}. \end{aligned}$$

Simple calculation show that (X^n, K^n) are defined by the following recurrent formula: $X_0^n = X_0, K_0^n = 0$,

$$\begin{cases} \Delta Y_{t_{k+1}}^n &= \varphi(f(Z_{t_{k+1}}^n - Z_{t_k}^n)) - X_{t_k}^n, \\ X_{t_{k+1}}^n &= \Pi_{\bar{D}}(X_{t_k}^n + \Delta Y_{t_{k+1}}^n) = \Pi_{\bar{D}}(\varphi(f(Z_{t_{k+1}}^n - Z_{t_k}^n))), \\ K_{t_{k+1}}^n &= K_{t_k}^n + (X_{t_{k+1}}^n - X_{t_k}^n) - \Delta Y_{t_{k+1}}^n \end{cases} \quad (3.8)$$

and $X_t^n = X_{t_k^n}^n$, $K_t^n = K_{t_k^n}^n$, $t \in [t_k^n, t_{k+1}^n)$, $k \in \mathbb{N} \cup \{0\}$. Since $Z^{\rho^n} \rightarrow Z$ a.s. in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$, the result is an immediate consequence of Theorem 3.1(i). \square

Theorem 3.3 *Assume (A), (B) and (F), (Δ) . Let Z be an (\mathcal{F}_t) adapted semimartingale and X_0 be a \mathcal{F}_0 measurable and such that $X_0 \in \bar{D}$. If $f, f'f$ are locally Lipschitz continuous, then there exists a unique strong solution of the SDE (2.6).*

PROOF. Let $\{\{\tau_k^n\}\}$ be an array of (\mathcal{F}_t) stopping times such that $\tau_0^n = 0$, $\tau_k^n = \inf\{t > \tau_{k-1}^n; |\Delta Z_t| > \frac{1}{n}\} \wedge (\tau_{k-1}^n + \frac{1}{n})$, $n, k \in \mathbb{N}$. Let us consider the sequence $\{Z^n\}$ of (\mathcal{F}_t) adapted semimartingales of the form: $Z_0^n = 0$ and

$$Z_t^n = Z_{\tau_k^n}^n, \quad t \in [\tau_k^n, \tau_{k+1}^n), \quad n, k \in \mathbb{N}.$$

It is easy to see that

$$\sup_{t \leq q} |Z_t^n - Z_t| \rightarrow 0, \quad P\text{-a.s.}, \quad q \in \mathbb{R}^+. \quad (3.9)$$

Observe that Z^n is an (\mathcal{F}_t) adapted semimartingale and the solution (X^n, K^n) of (3.1) driven by Z^n has the following form: $X_0^n = X_0$, $K_0^n = 0$,

$$\begin{cases} \Delta Y_{\tau_{k+1}^n}^n &= \varphi(f(Z_{\tau_{k+1}^n}^n - Z_{\tau_k^n}^n)) - X_{\tau_k^n}^n, \\ X_{\tau_{k+1}^n}^n &= \Pi_{\bar{D}}(\varphi(f(Z_{\tau_{k+1}^n}^n - Z_{\tau_k^n}^n))), \\ K_{\tau_{k+1}^n}^n &= K_{\tau_k^n}^n + (X_{\tau_{k+1}^n}^n - X_{\tau_k^n}^n) - \Delta Y_{\tau_{k+1}^n}^n \end{cases} \quad (3.10)$$

and $X_t^n = X_{\tau_k^n}^n$, $K_t^n = K_{\tau_k^n}^n$, $t \in [\tau_k^n, \tau_{k+1}^n)$, $k \in \mathbb{N} \cup \{0\}$. We will show that $\{(X^n, K^n)\}$ converges in probability. For this purpose it suffices to show that from any subsequences $(l) \subset (n)$, $(m) \subset (n)$ it is possible to choose further subsequences $(l_k) \subset (l)$, $(m_k) \subset (m)$ such that

$$((X^{l_k}, K^{l_k}), (X^{m_k}, K^{m_k})) \xrightarrow{\mathcal{D}} ((\hat{X}, \hat{K}), (\hat{X}', \hat{K}')) \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{4d})$$

(see Gyöngy and Krylov [5]). Using (3.9), the fact that $\{Z^n\}$ satisfies (UT) and arguments from the proof of Theorem 3.1(i) we show that

$$\{(X^l, K^l, Z^l, X^m, K^m, Z^m)\} \quad \text{is tight in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{6d}).$$

Therefore we can choose subsequences $(l_k) \subset (l)$, $(m_k) \subset (m)$ such that

$$(X^{l_k}, K^{l_k}, Z^{l_k}, X^{m_k}, K^{m_k}, Z^{m_k}) \xrightarrow{\mathcal{D}} (\hat{X}, \hat{K}, \hat{Z}, \hat{X}', \hat{K}', \hat{Z}), \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{6d}),$$

where \hat{Z} is a semimartingale with respect to the natural filtration $\mathcal{F}^{\hat{X}, \hat{K}, \hat{X}', \hat{K}', \hat{Z}}$ such that $\mathcal{L}(\hat{X}_0, \hat{X}) = \mathcal{L}(X_0, Z)$. By the arguments from the proof of Theorem 3.1(i) the processes (\hat{X}, \hat{K}) and (\hat{X}', \hat{K}') are two solutions of (2.6) with \hat{X}_0, \hat{Z} instead of X_0, Z . Since by Lemma 2.5 the SDE (2.6) is pathwise unique, $(\hat{X}, \hat{K}) = (\hat{X}', \hat{K}')$. Consequently, $\{(X^n, K^n)\}$ converges in probability in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d})$ to some pair of processes (X, K) . Clearly, (X, K) is a strong solution of (2.6), so using once again the pathwise uniqueness property we conclude that (X, K) is a unique strong solution of (2.6). \square

Theorem 3.4 *Assume (A), (B) and (F), (Δ) . Let $\{Z^n\}$ be a sequence of (\mathcal{F}_t^n) adapted semimartingales satisfying (UT) and $\{X_0^n\}$ be a sequence of \mathcal{F}_0^n measurable random variables such that $X_0^n \in \bar{D}$, $n \in \mathbb{N}$. If $\{(X^n, K^n)\}$ is a sequence of strong solutions of the SDE (3.1) and $f, f'f$ are locally Lipschitz continuous, then the following two implications are true:*

(i) if $X_0^n \xrightarrow{\mathcal{P}} X_0$ and $Z^n \xrightarrow{\mathcal{P}} Z$ in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ then

$$(X^n, K^n, Z^n) \xrightarrow{\mathcal{P}} (X, K, Z) \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d}),$$

(ii) if $X_0^n \xrightarrow{\mathcal{P}} X_0$ and $\sup_{t \leq q} |Z_t^n - Z_t| \xrightarrow{\mathcal{P}} 0$, $q \in \mathbb{R}^+$, then

$$\sup_{t \leq q} (|X_t^n - X_t| + |K_t^n - K_t|) \xrightarrow{\mathcal{P}} 0, \quad q \in \mathbb{R}^+,$$

where (X, K) is the unique strong solution of (2.6).

PROOF. (i) We use Theorem 3.3 and follow the proof of [27, Corollary 11(i)].

(ii) Let us note that

$$\Delta X_t = \varphi(\Delta Z_t f, X_{t-}) - X_{t-} + \Delta K_t$$

and if $\Delta X_t \neq 0$ then $\Delta Z_t \neq 0$. By part (i) and [26, Corollary C],

$$\sup_{t \leq q} |X_t^n - X_t| \xrightarrow{\mathcal{P}} 0, \quad q \in \mathbb{R}^+,$$

and the proof is complete. \square

Corollary 3.5 Assume (A), (B) and (F), (Δ) . Let Z be an (\mathcal{F}_t) adapted semimartingale and X_0 be a \mathcal{F}_0 measurable and such that $X_0 \in \bar{D}$. Let $\{T_n\}$ be a sequence of partitions of \mathbb{R}^+ satisfying (2.1) and $\{(X^n, K^n)\}$ denotes a sequence of solutions of (3.1) corresponding to $Z^n = Z^{\rho^n}$, $X_0^n = X_0$, $n \in \mathbb{N}$. If $f, f'f$ are locally Lipschitz continuous, then

$$\sup_{t \leq q} (|X_t^n - X_t^{\rho^n}| + |K_t^n - K_t^{\rho^n}|) \xrightarrow{\mathcal{P}} 0, \quad q \in \mathbb{R}^+,$$

where (X, K) is a unique strong solution of (2.6).

PROOF. By Theorem 3.4(i),

$$(X^n, K^n, Z^{\rho^n}) \xrightarrow{\mathcal{P}} (X, K, Z) \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{3d}).$$

Consequently,

$$(X^n, X^{\rho^n}) \xrightarrow{\mathcal{P}} (X, X) \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d}) \quad \text{and} \quad (K^n, K^{\rho^n}) \xrightarrow{\mathcal{P}} (K, K) \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d}),$$

which implies the corollary. \square

4 Approximations of Wong-Zakai type

Let us consider a sequence $\{T_n\}$ of partitions of \mathbb{R}^+ satisfying condition (2.1). We will approximate in probability the first coordinate X of the solution of (2.6) by solutions of nonreflected SDEs given by the following recurrent scheme: $\hat{X}_0^n = X_0$,

$$\hat{X}_t^n = \Pi_{\bar{D}}(\hat{X}_{t_k^n-}^n) + (t_{k+1}^n - t_k^n)^{-1} \int_{t_k^n}^t f(\hat{X}_s^n) ds (Z_{t_{k+1}^n}^n - Z_{t_k^n}^n), \quad t \in [t_k^n, t_{k+1}^n), \quad k \in \mathbb{N} \cup \{0\}.$$

One can observe that if \bar{Z}^n is the linear approximation of Z of the form $\bar{Z}_0^n = Z_0 = 0$,

$$\bar{Z}_t^n = Z_{t_k^n} + \frac{t - t_k^n}{t_{k+1}^n - t_k^n} (Z_{t_{k+1}^n} - Z_{t_k^n}), \quad t \in [t_k^n, t_{k+1}^n), \quad n \in \mathbb{N}, k \in \mathbb{N} \cup \{0\},$$

then \hat{X}^n satisfies the equation

$$\hat{X}_t^n = X_0 + \int_0^t f(\hat{X}_s^n) d\bar{Z}_s^n + \hat{K}_t^n, \quad n \in \mathbb{N},$$

where $\hat{K}_t^n = \sum_{k; t_k^n \leq t} (\Pi_{\bar{D}}(\hat{X}_{t_k^n-}^n) - \hat{X}_{t_k^n-}^n)$, $n \in \mathbb{N}$. Moreover, if (X^n, K^n) is the strong solution of (3.1) corresponding to Z^{ρ^n} then $\hat{X}_t^n = X_t^n$ for $t \in T_n$ and $\hat{K}_t^n = K_t^n$, $t \in \mathbb{R}^+$.

Theorem 4.1 *Assume (A), (B) and (F), (Δ) . If $f, f'f$ are locally Lipschitz continuous, then*

$$(i) \quad \sup_{t \leq q, t \in T_n} |\hat{X}_t^n - X_t| \xrightarrow{\mathcal{P}} 0, \quad q \in \mathbb{R}^+ \text{ and}$$

$$\hat{X}_t^n \xrightarrow{\mathcal{P}} X_t$$

provided that $\Delta Z_t = 0$ or $t \in \liminf_{n \rightarrow +\infty} T_n$,

$$(ii) \quad \hat{X}^n \xrightarrow{\mathcal{P}} X \text{ in the } S\text{-topology,}$$

(iii) *if moreover Z is a semimartingale with continuous trajectories then*

$$\sup_{t \leq q} |\hat{X}_t^n - X_t| \xrightarrow{\mathcal{P}} 0, \quad q \in \mathbb{R}^+,$$

where X is the first coordinate of the unique strong solution of the SDE (2.6).

PROOF. (i) Assume that $\{(X^n, K^n)\}$ is a family of strong solutions of (3.1) corresponding to $\{Z^{\rho^n}\}$. Since $\hat{X}^{n, \rho^n} = X^n$, it follows from Corollary (3.5) that

$$\sup_{t \leq q, t \in T_n} |\hat{X}_t^n - X_t| = \sup_{t \leq q} |\hat{X}_t^{n, \rho^n} - X_t^{\rho^n}| \xrightarrow{\mathcal{P}} 0, \quad q \in \mathbb{R}^+.$$

Moreover, if $\Delta Z_t = 0$ then $X_t^{\rho^n} \rightarrow X_t$. Therefore for $q \geq t$ and $t_k^n = \rho_n(t) \leq t$ we have

$$|\hat{X}_t^n - X_t| \leq \sup_{t \leq q, t \in T_n} |\hat{X}_t^n - X_t| + L|Z_{t_{k+1}^n} - Z_{t_k^n}| + |X_{t_k^n}^{\rho^n} - X_t| \rightarrow 0, \quad P\text{-a.s.},$$

where $L = \sup_{x \in \bar{D}} \|f(x)\|$.

(ii) We will show that $\{\hat{X}^n\}$ is tight in the S topology. By Theorem 3.1(i) the sequence $\{|\hat{K}^n|_q = |K^n|_q\}$ is bounded in probability. Therefore it suffices to show that

$$\{\hat{Y}^n = \int_0^\cdot f(\hat{X}_s^n) d\bar{Z}_s^n\} \text{ satisfies (UT)}$$

and then use [7, Theorem 4.1]. Set $\rho_t^{n,*} = \min\{t_k^n; t_k^n > t\}$ and $\hat{\mathcal{F}}_t^n = \mathcal{F}_{\rho_t^{n,*}} = \mathcal{F}_{t_k^n}^n$, $t \in [t_{k-1}^n, t_k^n)$, $k \in \mathbb{N} \cup \{0\}$, $n \in \mathbb{N}$. Clearly, \hat{Y}^n is an $(\hat{\mathcal{F}}_t^n)$ adapted processes admitting the decomposition into the sum of two $(\hat{\mathcal{F}}_t^n)$ adapted processes of the form

$$\int_0^\cdot f(\hat{X}_{s-}^{n,\rho^n}) d\bar{Z}_s^n + \int_0^\cdot (f(\hat{X}_s^n) - f(\hat{X}_{s-}^{n,\rho^n})) d\bar{Z}_s^n = I^{n,1} + I^{n,2}.$$

Since

$$I_t^{n,1} = \sum_{k; t_k^n \leq t} f(\hat{X}_{t_{k-1}^n}^n)(Z_{t_k^n} - Z_{t_{k-1}^n}) + (t - \rho_t^n) f(\hat{X}_{\rho_t^n}^n)(Z_{\rho_t^{n,*}} - Z_{\rho_t^n}), \quad t \in \mathbb{R}^+,$$

it follows from the Bichteler-Mokobodski theorem that $\{I^{n,1}\}$ satisfies (UT). On the other hand, in view of boundedness of $f'f$ there is $C > 0$ such that for all sufficiently large n ,

$$\begin{aligned} |I^{n,2}|_q &\leq \int_0^{\rho_q^{n,*}} \|f(\hat{X}_s^n) - f(\hat{X}_{s-}^{n,\rho^n})\| d|\bar{Z}|_s \\ &= \sum_{k; t_k^n \leq q} \int_{t_{k-1}^n}^{t_k^n} \|f(\hat{X}_s^n) - f(\hat{X}_{t_{k-1}^n}^n)\| d|\bar{Z}|_s \\ &\leq \sum_{k; t_k^n \leq \rho_q^{n,*}} \int_{t_{k-1}^n}^{t_k^n} \int_{t_{k-1}^n}^s \|f'f(\hat{X}_u^n)\| d|\bar{Z}|_u d|\bar{Z}|_s \\ &\leq C \sum_{k; t_k^n \leq \rho_q^{n,*}} |Z_{t_k^n} - Z_{t_{k-1}^n}|^2 \leq C \sum_{k; t_k^n \leq q+1} |Z_{t_k^n} - Z_{t_{k-1}^n}|^2. \end{aligned}$$

This implies that $\{|I^{n,2}|_q\}$ is bounded in probability and completes the proof of (ii).

(iii) In this case the solution X has continuous trajectories as well. Therefore $\sup_{t \leq q} |X_t - X_t^{\rho^n}| \rightarrow 0$, P -a.s., $q \in \mathbb{R}^+$. Similarly, since the trajectories of Z are continuous, we have $\sup_{t \leq q} |\Delta Z_t^{\rho^n}| \rightarrow 0$, P -a.s., $q \in \mathbb{R}^+$, which implies that for all sufficiently large n ,

$$\sup_{t \leq q} |\hat{X}_t^n - \hat{X}_t^{n,\rho^n}| \leq L \sup_{t \leq q+1} |\Delta Z_t^{\rho^n}| \rightarrow 0, \quad P\text{-a.s.} \quad (4.1)$$

Now, (iii) is an easy consequence of (i). \square

We now consider standard Wong-Zakai type approximation of reflected Stratonovich SDEs. The approximation processes (\bar{X}^n, \bar{K}^n) are defined by the recurrent scheme: $\bar{X}_0^n = X_0$ and

$$\bar{X}_t^n = \bar{X}_{t_k^n}^n + (t_{k+1}^n - t_k^n)^{-1} \int_{t_k^n}^t f(\bar{X}_s^n) ds (Z_{t_{k+1}^n} - Z_{t_k^n}) + \bar{K}_t^n - \bar{K}_{t_k^n}^n \quad (4.2)$$

for $t \in [t_k^n, t_{k+1}^n)$, $k \in \mathbb{N} \cup \{0\}$. Clearly, (\bar{X}^n, \bar{K}^n) is a solution of the Skorokhod problem associated with

$$\bar{Y}^n = X_0 + \int_0^\cdot f(\bar{X}_s^n) d\bar{Z}_s^n, \quad n \in \mathbb{N}.$$

Theorem 4.2 Assume (A), (B) and (F). If $f, f'f$ are locally Lipschitz continuous and Z is a semimartingale with continuous trajectories then

$$\sup_{t \leq q} (|\bar{X}_t^n - X_t| + |\bar{K}_t^n - K_t|) \xrightarrow{P} 0, \quad q \in \mathbb{R}^+.$$

PROOF. We will use the notation from the proof of Theorem 4.1. For brevity, we write $\Delta Z_k = Z_{t_k^n} - Z_{t_{k-1}^n}$, $\Delta t_k = (t_k^n - t_{k-1}^n)$, $k \in \mathbb{N}$. From Theorem 4.1, we know that $\sup_{t \leq q} |\hat{X}_t^n - X_t| \rightarrow_{\mathcal{P}} 0$, $q \in \mathbb{R}^+$. Moreover, $(\hat{X}^{n,\rho^n}, \hat{K}^n) = (\hat{X}^{n,\rho^n}, \hat{K}^{n,\rho^n}) = (X^n, K^n)$ is a solution of the Skorokhod problem associated with $Y^n = X_0 + \int_0^{\rho^n} f(\hat{X}_s^n) d\bar{Z}_s^n$ and

$$|\Delta K_t^n| \leq |\Delta Y_t^n| \leq \sup_x \|f(x)\| |\Delta Z_t^{\rho^n}|, \quad t \in \mathbb{R}^+. \quad (4.3)$$

Since $\{Y^n = X^n - K^n\}$ converges uniformly in probability, it follows from [27, Proposition 3] that

$$\{|K^n|_q\} \text{ is bounded in probability } q \in \mathbb{R}^+. \quad (4.4)$$

In the rest of the proof we will show that

$$\sup_{t \leq q} |\hat{X}_t^n - \bar{X}_t^n| \xrightarrow{\mathcal{P}} 0, \quad q \in \mathbb{R}^+. \quad (4.5)$$

We start with proving that (4.4) holds with K^n replaced by \bar{K}^n . To this end, we decompose Y^n into the sum

$$\begin{aligned} \bar{Y}_t^n &= (\bar{Y}_t^n - \bar{Y}_t^{n,\rho^n}) + \int_0^{\rho_t^n} f(\bar{X}_{s-}^{n,\rho^n}) dZ_s^{\rho^n} + \int_0^{\rho_t^n} (f(\bar{X}_s^n) - f(\bar{X}_{s-}^{n,\rho^n})) d\bar{Z}_s^n \\ &= I_t^{n,1} + I_t^{n,2} + I_t^{n,3}. \end{aligned}$$

Since $\{Z^{\rho^n}\}$ satisfies (UT) and $\sup_{t \leq q} |Z_t^{\rho^n} - Z_t| \rightarrow 0$, P -a.s., $q \in \mathbb{R}^+$, and since moreover f is bounded, we see that for all sufficiently large n ,

$$\sup_{t \leq q} |I_t^{n,1}| = \sup_{t \leq q} |\bar{Y}_t^n - \bar{Y}_t^{n,\rho^n}| \leq L \sup_{t \leq q+1} |\Delta Z_t^{\rho^n}| \rightarrow 0, \quad P\text{-a.s.}, \quad q \in \mathbb{R}^+ \quad (4.6)$$

and

$$\{I^{n,2} = \int_0^{\rho^n} f(\bar{X}_{s-}^{n,\rho^n}) dZ_s^{\rho^n}\} \text{ is C-tight.} \quad (4.7)$$

Furthermore,

$$\begin{aligned} I_t^{n,3} &= \int_0^{\rho_t^n} (f(\bar{X}_s^n) - f(\bar{X}_{s-}^{n,\rho^n})) d\bar{Z}_s^n = \sum_{k; t_k^n \leq t} \int_{t_{k-1}^n}^{t_k^n} (f(\bar{X}_s^n) - f(\bar{X}_{t_{k-1}^n}^n)) d\bar{Z}_s^n \\ &= \sum_{k; t_k^n \leq t} \int_{t_{k-1}^n}^{t_k^n} \int_{t_{k-1}^n}^s f'(\bar{X}_u^n) d\bar{X}_u^n d\bar{Z}_s^n. \end{aligned}$$

Since $f'f$ is bounded, it follows from Lemma 2.2 that there is $C > 0$ such that

$$\begin{aligned} \left| \int_{t_{k-1}^n}^s f'(\bar{X}_u^n) d\bar{X}_u^n \right| &\leq \int_{t_{k-1}^n}^s \|f'f(\bar{X}_u^n)\| du \left| \frac{\Delta Z_k}{\Delta t_k} \right| + \left| \int_{t_{k-1}^n}^s f'(\bar{X}_u^n) d\bar{K}_u^n \right| \\ &\leq 2 \int_{t_{k-1}^n}^s \|f'f(\bar{X}_u^n)\| du \left| \frac{\Delta Z_k}{\Delta t_k} \right| \leq C |\Delta Z_k|. \end{aligned}$$

It follows that for any $s < t$,

$$|I_t^{n,3} - I_s^{n,3}| \leq C \sum_{k; s < t_k^n \leq t} |Z_{t_k^n} - Z_{t_{k-1}^n}|^2 = C([Z^{\rho^n}]_t - [Z^{\rho^n}]_s).$$

From the above and the fact that $\sup_{t \leq q} |[Z^{\rho^n}]_t - [Z]_t| \xrightarrow{\mathcal{P}} 0$, $q \in \mathbb{R}^+$, we deduce that

$$\{I^{n,3} = \int_0^{\rho^n} (f(\bar{X}_s^n) - f(\bar{X}_{s-}^{n,\rho^n})) d\bar{Z}_s^n\} \text{ is C-tight.} \quad (4.8)$$

Combining (4.8) with (4.7) and (4.6) we see that $\{\bar{Y}^n = I^{n,1} + I^{n,2} + I^{n,3}\}$ is C-tight. Therefore, by [27, Proposition 3,4],

$$\{|\bar{K}^n|_q\} \text{ is bounded in probability} \quad (4.9)$$

and

$$\{(\bar{X}^n, \bar{K}^n)\} \text{ is C-tight.} \quad (4.10)$$

By (4.1) and (4.10), to prove (4.5) it suffices now to show that

$$\sup_{t \leq q} |\hat{X}_t^{n,\rho^n} - \bar{X}_t^{n,\rho^n}| \xrightarrow{\mathcal{P}} 0, \quad q \in \mathbb{R}^+. \quad (4.11)$$

To check this we will use the stochastic Gronwall inequality. Since Z is a continuous semimartingale, it admits decomposition $Z = M + V$, where M is a continuous locally square integrable martingale such that $M_0 = 0$ and V is a continuous predictable process with bounded variation such that $V_0 = 0$. Set

$$\sigma_a^n = \inf\{t; \max(|M_t|, |V_t|, |\bar{X}_t^n|, |\bar{K}_t^n|) > a\}, \quad a \in \mathbb{R}^+, n \in \mathbb{N}.$$

Obviously, $\lim_{a \rightarrow \infty} \limsup_{n \rightarrow \infty} P(\sigma_a^n \leq q) = 0$, $q \in \mathbb{R}^+$. Therefore in each step n we can restrict our attention to processes stopped at σ_a^n . Now, for $n \in \mathbb{N}$, $b > 0$ set

$$\tau_b^n = \inf\{t > 0; \max([M^{\rho^n}]_t, \langle M^{\rho^n} \rangle_t, |V^{\rho^n}|_t^2, |\hat{X}_t^{n,\rho^n}|, |\hat{K}_t^{n,\rho^n}|) > b\}.$$

The processes $[M^{\rho^n}]$, $\langle M^{\rho^n} \rangle$, $|V^{\rho^n}|^2$, $|\hat{X}_t^{n,\rho^n}|$, $|\hat{K}_t^{n,\rho^n}|$ stopped at τ_b^n are bounded. Moreover,

$$\lim_{b \rightarrow \infty} \limsup_{n \rightarrow \infty} P(\tau_b^n \leq q) = 0, \quad q \in \mathbb{R}^+,$$

so we can restrict our attention to the processes stopped at τ_b^n . By [25, Lemma 2.3(i)],

$$\begin{aligned} |\bar{X}_t^{n,\rho^n} - \hat{X}_t^{n,\rho^n}|^2 &\leq |\bar{Y}_t^{n,\rho^n} - \hat{Y}_t^{n,\rho^n}|^2 + \frac{1}{r_0} \int_0^{\rho_t^n} |\bar{X}_s^n - \hat{X}_s^{n,\rho^n}|^2 d(|\bar{K}^n| + |\hat{K}^{n,\rho^n}|)_s \\ &\quad + 2 \int_0^{\rho_t^n} (\bar{Y}_t^{n,\rho^n} - \bar{Y}_s^n) - (\hat{Y}_t^{n,\rho^n} - \hat{Y}_s^{n,\rho^n}) d(\bar{K}^n - \hat{K}^{n,\rho^n})_s \\ &= |\bar{Y}_t^{n,\rho^n} - \hat{Y}_t^{n,\rho^n}|^2 + \frac{1}{r_0} \int_0^t |\bar{X}_{s-}^{n,\rho^n} - \hat{X}_{s-}^{n,\rho^n}| d(|\bar{K}^{n,\rho^n}| + |\hat{K}^{n,\rho^n}|)_s \\ &\quad + 2 \int_0^t (\bar{Y}_t^{n,\rho^n} - \bar{Y}_s^{n,\rho^n}) - (\hat{Y}_t^{n,\rho^n} - \hat{Y}_s^{n,\rho^n}) d(\bar{K}^{n,\rho^n} - \hat{K}^{n,\rho^n})_s + R_t^{n,1}, \end{aligned}$$

where

$$\begin{aligned} R_t^{n,1} &= \frac{1}{r_0} \int_0^{\rho_t^n} |(\bar{X}_s^n - \bar{X}_{s-}^{n,\rho^n}) - (\hat{X}_s^{n,\rho^n} - \hat{X}_{s-}^{n,\rho^n})|^2 d(|\bar{K}^n| + |\hat{K}^{n,\rho^n}|)_s \\ &\quad + 2 \int_0^{\rho_t^n} (\bar{Y}_s^{n,\rho^n} - \bar{Y}_s^n) d(\bar{K}^n - \hat{K}^{n,\rho^n})_s. \end{aligned}$$

By the integration by parts formula,

$$\begin{aligned}
& 2 \int_0^t (\bar{Y}_t^{n,\rho^n} - \bar{Y}_s^{n,\rho^n}) - (\hat{Y}_t^{n,\rho^n} - \hat{Y}_s^{n,\rho^n}) d(\bar{K}^{n,\rho^n} - \hat{K}^{n,\rho^n})_s \\
&= 2 \int_0^t (\bar{K}_{s-}^{n,\rho^n} - \hat{K}_{s-}^{n,\rho^n}) d(\bar{Y}^{n,\rho^n} - \hat{Y}^{n,\rho^n})_s \\
&= 2 \int_0^t (\bar{X}_{s-}^{n,\rho^n} - \hat{X}_{s-}^{n,\rho^n}) d(\bar{Y}^{n,\rho^n} - \hat{Y}^{n,\rho^n})_s - 2 \int_0^t (\bar{Y}_{s-}^{n,\rho^n} - \hat{Y}_{s-}^{n,\rho^n}) d(\bar{Y}^{n,\rho^n} - \hat{Y}^{n,\rho^n})_s \\
&= 2 \int_0^t (\bar{X}_{s-}^{n,\rho^n} - \hat{X}_{s-}^{n,\rho^n}) d(\bar{Y}^{n,\rho^n} - \hat{Y}^{n,\rho^n})_s + [\bar{Y}^{n,\rho^n} - \hat{Y}^{n,\rho^n}]_t - |\bar{Y}_t^{n,\rho^n} - \hat{Y}_t^{n,\rho^n}|^2,
\end{aligned}$$

which implies that

$$\begin{aligned}
|\bar{X}_t^{n,\rho^n} - \hat{X}_t^{n,\rho^n}|^2 &\leq [\bar{Y}^{n,\rho^n} - \hat{Y}^{n,\rho^n}]_t + \frac{1}{r_0} \int_0^t |\bar{X}_{s-}^{n,\rho^n} - \hat{X}_{s-}^{n,\rho^n}| d(|\bar{K}^{n,\rho^n}| + |\hat{K}^{n,\rho^n}|)_s \\
&\quad + 2 \int_0^t (\bar{X}_{s-}^{n,\rho^n} - \hat{X}_{s-}^{n,\rho^n}) d(\bar{Y}^{n,\rho^n} - \hat{Y}^{n,\rho^n})_s + R_t^{n,1} \\
&= I_t^{n,1} + I_t^{n,2} + I_t^{n,3} + R_t^{n,1}.
\end{aligned} \tag{4.12}$$

Since we may assume that f is Lipschitz continuous, we have

$$\begin{aligned}
I_t^{n,1} &= \sum_{k; t_k^n \leq t} \left| \int_{t_{k-1}^n}^{t_k^n} (f(\bar{X}_s^n) - f(\hat{X}_s^n)) ds \frac{\Delta Z_k}{\Delta t_k} \right|^2 \\
&\leq C \sum_{k; t_k^n \leq t} |\bar{X}_{t_{k-1}^n}^n - \hat{X}_{t_{k-1}^n}^n|^2 |\Delta Z_k|^2 \\
&\quad + C \sup_{s \leq t} |(\bar{X}_s^n - \bar{X}_s^{n,\rho^n}) + (\hat{X}_s^n - \hat{X}_s^{n,\rho^n})|^2 \sum_{k; t_k^n \leq t} |\Delta Z_k|^2 \\
&= C \int_0^t |\bar{X}_{s-}^{n,\rho^n} - \hat{X}_{s-}^{n,\rho^n}|^2 d[Z^{\rho^n}]_s + R_t^{n,2}.
\end{aligned}$$

Similarly, by using Lipschitz continuity of $f'f$ we get

$$\begin{aligned}
|I_t^{n,3}| &= \left| \sum_{k; t_k^n \leq t} (\bar{X}_{t_{k-1}^n}^n - \hat{X}_{t_{k-1}^n}^n) \left(\int_{t_{k-1}^n}^{t_k^n} (f(\bar{X}_s^n) - f(\hat{X}_s^n)) ds \frac{\Delta Z_k}{\Delta t_k} \right) \right| \\
&\leq \left| \sum_{k; t_k^n \leq t} (\bar{X}_{t_{k-1}^n}^n - \hat{X}_{t_{k-1}^n}^n) (f(\bar{X}_{t_{k-1}^n}^n) - f(\hat{X}_{t_{k-1}^n}^n)) (\Delta Z_k) \right| \\
&\quad + \left| \sum_{k; t_k^n \leq t} (\bar{X}_{t_{k-1}^n}^n - \hat{X}_{t_{k-1}^n}^n) \left(\int_{t_{k-1}^n}^{t_k^n} (f'(\bar{X}_u^n) d\bar{X}_u^n - f'(\hat{X}_u^n) d\hat{X}_u^n) ds \frac{\Delta Z_k}{\Delta t_k} \right) \right| \\
&\leq \left| \int_0^t (\bar{X}_{s-}^{n,\rho^n} - \hat{X}_{s-}^{n,\rho^n}) (f(\bar{X}_{s-}^{n,\rho^n}) - f(\hat{X}_{s-}^{n,\rho^n})) dZ_s^{\rho^n} \right| \\
&\quad + C \int_0^t |\bar{X}_{s-}^{n,\rho^n} - \hat{X}_{s-}^{n,\rho^n}|^2 d[Z^{\rho^n}]_s + R_t^{n,3}
\end{aligned}$$

and

$$\begin{aligned}
|R_t^{n,3}| &\leq C \sup_{s \leq t} |(\bar{X}_s^n - \bar{X}_s^{n,\rho^n}) + (\hat{X}_s^n - \hat{X}_s^{n,\rho^n})| \sum_{k; t_k^n \leq t} |\bar{X}_{t_{k-1}}^n - \hat{X}_{t_{k-1}}^n| |\Delta Z_k|^2 \\
&\quad + \left| \sum_{k; t_k^n \leq t} (\bar{X}_{t_{k-1}}^n - \hat{X}_{t_{k-1}}^n) \left(\int_{t_{k-1}}^{t_k^n} \left(\int_{t_{k-1}}^s (f'(\bar{X}_u^n) d\bar{K}_u^n) ds \right) \frac{\Delta Z_k}{\Delta t_k} \right) \right| \\
&\leq C \sup_{s \leq t} |(\bar{X}_s^n - \bar{X}_s^{n,\rho^n}) + (\hat{X}_s^n - \hat{X}_s^{n,\rho^n})| \int_0^t |\bar{X}_{s-}^n - \hat{X}_{s-}^n| d[Z^{\rho^n}]_s \\
&\quad + C \max_{k; t_k^n \leq t} |\Delta Z_k| \int_0^t |\bar{X}_{s-}^{n,\rho^n} - \hat{X}_{s-}^{n,\rho^n}| d|\bar{K}^{n,\rho^n}|_s.
\end{aligned}$$

Substituting the last three estimates into (4.12) we obtain

$$\begin{aligned}
|\bar{X}_t^{n,\rho^n} - \hat{X}_t^{n,\rho^n}|^2 &\leq C \int_0^t |\bar{X}_{s-}^{n,\rho^n} - \hat{X}_{s-}^{n,\rho^n}|^2 d[Z^{\rho^n}]_s \\
&\quad + \frac{1}{r_0} \int_0^t |\bar{X}_{s-}^{n,\rho^n} - \hat{X}_{s-}^{n,\rho^n}| d(|\bar{K}^{n,\rho^n}| + |\hat{K}^{n,\rho^n}|)_s \\
&\quad + \left| \int_0^t (\bar{X}_{s-}^{n,\rho^n} - \hat{X}_{s-}^{n,\rho^n}) (f(\bar{X}_{s-}^{n,\rho^n}) - f(\hat{X}_{s-}^{n,\rho^n})) dZ_s^{\rho^n} \right| + R_t^n,
\end{aligned} \tag{4.13}$$

where $R_t^n = R_t^{n,1} + R_t^{n,2} + R_t^{n,3}$, $t \in \mathbb{R}^+$, $n \in \mathbb{N}$. Simple calculation shows that for any $q \in \mathbb{R}^+$, $\epsilon_n = E \sup_{t \leq q} |R_t^n| \rightarrow 0$. Fix $q \in \mathbb{R}^+$. Later on we will restrict our attention to processes which in addition are stopped at q . Set $A_t^n = [Z^{\rho^n}]_t + |V^{\rho^n}|_t + |\bar{K}^{n,\rho^n}|_t/r_0 + |\hat{K}^{n,\rho^n}|_t/r_0$. Clearly, there is $C_1 > 0$ such that $A_\infty^n + [M^{\rho^n}]_\infty + \langle M^{\rho^n} \rangle_\infty \leq C_1$. On the other hand, by (4.13), there is $C_2 > 0$ such that for any $(\mathcal{F}_t^{\rho^n})$ stopping time γ^n ,

$$\begin{aligned}
E \sup_{t < \gamma^n} |\bar{X}_t^{n,\rho^n} - \hat{X}_t^{n,\rho^n}|^2 &\leq C_2 \int_0^{\gamma^n-} \sup_{u \leq s} |\bar{X}_{u-}^{n,\rho^n} - \hat{X}_{u-}^{n,\rho^n}|^2 dA_s^n \\
&\quad + E \left(\sup_{t \leq \gamma^n-} \left| \int_0^t (\bar{X}_{s-}^{n,\rho^n} - \hat{X}_{s-}^{n,\rho^n}) (f(\bar{X}_{s-}^{n,\rho^n}) - f(\hat{X}_{s-}^{n,\rho^n})) dM_s^{\rho^n} \right| \right) + \epsilon_n.
\end{aligned}$$

Since f is Lipschitz continuous, it follows from the version of Metivier-Pellaumail inequality proved in Pratelli [22] and Schwartz inequality that

$$\begin{aligned}
&E \left(\sup_{t < \gamma^n} \left| \int_0^t (\bar{X}_{s-}^{n,\rho^n} - \hat{X}_{s-}^{n,\rho^n}) (f(\bar{X}_{s-}^{n,\rho^n}) - f(\hat{X}_{s-}^{n,\rho^n})) dM_s^{\rho^n} \right| \right) \\
&\leq cE \left(\int_0^{\gamma^n-} |\bar{X}_{s-}^{n,\rho^n} - \hat{X}_{s-}^{n,\rho^n}|^4 d([M^{\rho^n}] + \langle M^{\rho^n} \rangle)_s \right)^{1/2} \\
&\leq cE \sup_{t < \gamma^n} |\bar{X}_{s-}^{n,\rho^n} - \hat{X}_{s-}^{n,\rho^n}| \left(\int_0^{\gamma^n-} |\bar{X}_{s-}^{n,\rho^n} - \hat{X}_{s-}^{n,\rho^n}|^2 d([M^{\rho^n}] + \langle M^{\rho^n} \rangle)_s \right)^{1/2} \\
&\leq c(E \sup_{s < \gamma^n} |\bar{X}_{s-}^{n,\rho^n} - \hat{X}_{s-}^{n,\rho^n}|^2)^{1/2} (E \int_0^{\gamma^n-} |\bar{X}_{s-}^{n,\rho^n} - \hat{X}_{s-}^{n,\rho^n}|^2 d([M^{\rho^n}] + \langle M^{\rho^n} \rangle)_s)^{1/2} \\
&\leq \frac{1}{2} E \sup_{s < \gamma^n} |\bar{X}_{s-}^{n,\rho^n} - \hat{X}_{s-}^{n,\rho^n}|^2 + c' E \int_0^{\gamma^n-} |\bar{X}_{s-}^{n,\rho^n} - \hat{X}_{s-}^{n,\rho^n}|^2 d([M^{\rho^n}] + \langle M^{\rho^n} \rangle)_s.
\end{aligned}$$

Therefore, for any $(\mathcal{F}_t^{\rho^n})$ stopping time γ^n ,

$$\begin{aligned} E \sup_{t < \gamma^n} |\bar{X}_t^{n, \rho^n} - \hat{X}_t^{n, \rho^n}|^2 \\ \leq (2C_2 + 2c')E \int_0^{\gamma^n -} \sup_{u \leq s} |\bar{X}_{u-}^{n, \rho^n} - \hat{X}_{u-}^{n, \rho^n}|^2 d(A^n + [M^{\rho^n}] + \langle M^{\rho^n} \rangle)_s + 2\epsilon_n. \end{aligned}$$

From the stochastic version of Gronwall's lemma (see, e.g., [15, Lemma 2] or [29, Lemma C1]) it now follows that

$$E \sup_{t < q \wedge \tau_b^n} |\bar{X}_t^{n, \rho^n} - \hat{X}_t^{n, \rho^n}|^2 \leq 2\epsilon_n \exp\{(2C_2 + 2c')C_1\} \longrightarrow 0.$$

Hence we conclude (4.11) and completes the proof. \square

Remark 4.3 In general, if Z is discontinuous, the solutions of (2.6) cannot be approximated by solutions of (4.2). To see this, let us consider Z such that $Z_t = 0$ if $t < 1$ and $Z_t = 1$, otherwise. Set $t_k^n = k/n$, $n \in \mathbb{N}$, $k \in \mathbb{N} \cup \{0\}$. If f is bounded and Lipschitz continuous then there exists a unique solution (\bar{X}^n, \bar{K}^n) of (4.2). Moreover, it is easy to check that $\bar{X}_t^n \rightarrow X_0$ if $t < 1$ and $\bar{X}_t^n \rightarrow X_1$ if $t \geq 1$, where (X, K) is a solution of the reflecting equation

$$X_t = X_0 + \int_0^t f(X_s) ds \Delta Z_1 + K_t, \quad t \in \mathbb{R}^+.$$

However, the limit of $\{\bar{X}^n\}$ need not be a solution of the Skorokhod problem with jumps (in general we do not have the property $X_1 = \Pi_{\bar{D}}(X_0 + \int_0^1 f(X_s) ds \Delta Z_1)$). Consequently, the limit need not be a solution of (2.6) driven by Z .

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References

- [1] S. Aida, K. Sasaki, Wong-Zakai approximation of solutions to reflecting stochastic differential equations on domains in Euclidean spaces, *Stochastic Process. Appl.* 123 (2013) 3800–3827.
- [2] H. Doss, P. Priouret, Support d'un processus de reflexion, *Z. Wahrsch. Verw. Gebiete* 61 (3) (1982) 327–345.
- [3] S.N. Ethier, T.G. Kurtz, *Markov Processes*, Wiley, New York 1986.
- [4] L.C. Evans, D.W. Stroock, An approximation scheme for reflected stochastic differential equations, *Stochastic Process. Appl.* 121 (2011) 1464–1491.
- [5] I. Gyöngy, N. Krylov, Existence of strong solutions for Itô stochastic equations via approximations, *Probab. Theory Related Fields* 105 (1996) 143–158.
- [6] J. Jacod, *Convergence en loi de semimartingales et variation quadratique*, *Lecture Notes in Math* 721 Springer–Verlag 1979.

- [7] A. Jakubowski, A non-Skorohod topology on the Skorohod space, *EJP* 2 (1997) 1-21.
- [8] A. Jakubowski, J. Mémin, G. Pages, Convergence en loi des suites d'intégrales stochastiques sur l'espace D^1 de Skorokhod, *Probab. Theory Related Fields* 81 (1989) 111–137.
- [9] A. Kohatsu-Higa, Stratonovich type SDEs with normal reflection driven by semimartingales, *Sankhya* 63 A (2) (2001), 194-228.
- [10] T.G. Kurtz, Random time changes and convergence in distribution under the Meyer–Zheng conditions, *Ann. Probab.* 19 (1991) 1010–1034.
- [11] T.G. Kurtz, P. Protter, Weak limit theorems for stochastic integrals and stochastic differential equations, *Ann. Probab.* 19 (1991), 1035–1070.
- [12] T.G. Kurtz, P. Protter, Wong–Zakai corrections, random evolutions and simulation schemes for SDE's, *Proc. Conference in Honor Moshe Zakai 65th Birthday, Haifa, Stochastic Analysis* (1991), 331–346.
- [13] T.G. Kurtz, E. Pardoux, P. Protter, Stratonovich stochastic differential equations driven by general semimartingales, *Ann. Inst. Henri Poincaré* 31 (2) (1995) 351–377.
- [14] P.L. Lions, A.S. Sznitman, Stochastic Differential Equations with Reflecting Boundary Conditions, *Comm. Pure and Appl. Math.* XXXVII (1983) 511–537.
- [15] V. Mackievicius, \mathcal{S}^p stability of symmetric stochastic differential equations with discontinuous driving semimartingales, *Ann. Inst. Henri Poincaré* B 23 (1987) 575–592.
- [16] S. Marcus, Modeling and analysis of stochastic differential equations driven by point processes, *IEEE Transaction on Information Theory* 24 (1978), 164–172.
- [17] S. Marcus, Modeling and approximation of stochastic differential equations driven by semimartingales, *Stochastics* 4 (1981) 223–245.
- [18] J. Mémin, L. Słomiński, Condition UT et stabilité en loi des solutions d'équations différentielles stochastiques, *Sém. de Probabilités XXV, Lecture Notes in Math.* 1485 Springer–Verlag, Berlin Heidelberg New York 1991 162–177.
- [19] P.A. Meyer and W.A. Zheng, Tightness criteria for laws of semimartingales, *Ann. Inst. Henri Poincaré* B 20 (1984) 353–372.
- [20] P.A. Meyer, Un cours sur intégrales stochastiques. *Sém. de Probabilités X Lecture Notes in Math.* 511 Springer–Verlag, Berlin Heidelberg New York 1976.
- [21] R. Petterson, Wong–Zakai approximations for reflecting stochastic differential equations, *Stoch. Anal. Appl.* 17 (4) (1999) 609–617.
- [22] M. Pratelli, Majoration dans L^p du type Metivier-Pellaumail pour les semimartingales, *Sem. de Probab. XVII Lect. Notes in Math.* 986 Springer New York (1983) 125–131.
- [23] J. Ren, S. Xu, A transfer principle for multivalued stochastic differential equations, *J. Funct. Anal.* 256 (9) (2009) 2780–2814.

- [24] J. Ren, S. Xu, Support theorem for stochastic variational inequalities, *Bull. Sci Math.* 134 (8) (2010) 826–856.
- [25] Y. Saisho, Stochastic differential equations for multi-dimensional domain with reflecting boundary, *Probab. Theory Related Fields* 74 (1987) 455–477.
- [26] L. Słomiński, Stability of strong solutions of stochastic differential equations, *Stochastic Process. Appl.* 31 (1989) 173–202.
- [27] L. Słomiński, On existence, uniqueness and stability of solutions of multidimensional SDE's with reflecting boundary conditions, *Ann. Inst. H. Poincaré* 29.2 (1993) 163–198.
- [28] L. Słomiński, On approximation of solutions of multidimensional SDE's with reflecting boundary conditions, *Stoch. Process. Appl.* 50 (1994) 197–219.
- [29] L. Słomiński, Stability of stochastic differential equations driven by general semimartingales. *Diss. Math.* CCCXLIX (1996) 1–113.
- [30] C. Stricker, Loi de semimartingales et critères de compacité, *Sém. de Probab. XIX Lect. Notes in Math.* 1123 Springer–Verlag, Berlin Heidelberg New York 1985.
- [31] H. Tanaka, Stochastic differential equations with reflecting boundary condition in convex regions, *Hiroshima Math. J.* 9 (1979) 163–177.
- [32] E. Wong, M. Zakai, On the convergence of ordinary integrals to stochastic integrals, *Ann. Math. Statist.* 36 (1965) 1560–1564.
- [33] E. Wong, M. Zakai, On the relation between ordinary and stochastic differential equations, *Internat. J. Energ. Sci.* 3 (1965) 213–229.
- [34] T. Zhang, On the strong solutions of one-dimensional differential equations with reflecting boundary, *Stochastic Process. Appl.* 50 (1994) 135–147.
- [35] T. Zhang, Strong Convergence of Wong-Zakai Approximations of Reflected SDEs in a Multidimensional General Domain, *Potential Anal.* 41 (2014), 783–815.